

Kernel estimates for Schrödinger type operators with unbounded diffusion and potential terms

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Abstract

We prove that the heat kernel associated to the Schrödinger type operator $A := (1 + |x|^\alpha)\Delta - |x|^\beta$ satisfies the estimate

$$k(t, x, y) \leq c_1 e^{\lambda_0 t} e^{c_2 t^{-b}} \frac{(|x||y|)^{-\frac{N-1}{2} - \frac{\beta-\alpha}{4}}}{1 + |y|^\alpha} e^{-\frac{2}{\beta-\alpha+2}|x|^{\frac{\beta-\alpha+2}{2}}} e^{-\frac{2}{\beta-\alpha+2}|y|^{\frac{\beta-\alpha+2}{2}}}$$

for $t > 0$, $|x|, |y| \geq 1$, where c_1, c_2 are positive constants and $b = \frac{\beta-\alpha+2}{\beta+\alpha-2}$ provided that $N > 2$, $\alpha \geq 2$ and $\beta > \alpha - 2$. We also obtain an estimate of the eigenfunctions of A .

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1 Introduction

In this paper we consider the operator

$$Au(x) = (1 + |x|^\alpha)\Delta u(x) - |x|^\beta u(x), \quad x \in \mathbb{R}^N,$$

for $N > 2$, $\alpha \geq 2$ and $\beta > \alpha - 2$. We propose to study the behaviour of the associated heat kernel and associated eigenfunctions.

Recently several paper have dealt with elliptic operators with polynomially growing diffusion coefficients (see for example [13], [14], [15], [16], [11], [7], [3], [10], [9], [4], [6]).

In [11] (resp. [3]) it is proved that the realization A_p of A in $L^p(\mathbb{R}^N)$ for $1 < p < \infty$ with domain

$$D_p(A) = \{u \in W^{2,p}(\mathbb{R}^N) \mid (1 + |x|^\alpha)|D^2 u|, (1 + |x|^\alpha)^{1/2} \nabla u, |x|^\beta u \in L^p(\mathbb{R}^N)\}$$

generates a strongly continuous and analytic semigroup $T_p(\cdot)$ for $\alpha \in [0, 2]$ and $\beta > 0$ (resp. $\alpha > 2$ and $\beta > \alpha - 2$). This semigroup is also consistent, irreducible and ultracontractive. For the case $\beta = 0$ we refer to [7] and [13].

Since the coefficients of the operator A are locally regular it follows that the semigroup $T_p(\cdot)$ admits an integral representation through a heat kernel $k(t, x, y)$

$$T_p(t)u(x) = \int_{\mathbb{R}^N} k(t, x, y)u(y)dy, \quad t > 0, x \in \mathbb{R}^N,$$

for all $u \in L^p(\mathbb{R}^N)$ (cf. [2], [12]).

In [11] estimates of the kernel $k(t, x, y)$ for $\alpha \in [0, 2)$ and $\beta > 2$ were obtained. Our contribution in this paper is to show similar upper bounds for the case $\alpha \geq 2$ and $\beta > \alpha - 2$. Our techniques consist in providing upper and lower estimates for the ground state of A_p corresponding to the largest eigenvalue λ_0 and adapting the arguments used in [5].

The paper is structured as follows. In section 2 we prove that the eigenfunction $\psi(x)$ associated to the largest eigenvalue λ_0 can be estimated from below and above by the function

$$|x|^{-\frac{N-1}{2}-\frac{\beta-\alpha}{4}} e^{-\frac{2}{\beta-\alpha+2}|x|^{\frac{\beta-\alpha+2}{2}}}$$

for $|x| \geq 1$.

In Section 3 we introduce the measure $d\mu = (1 + |x|^\alpha)^{-1}dx$ for which the operator A is symmetric and generates an analytic semigroup (which is a Markov semigroup) with kernel

$$k_\mu(t, x, y) = (1 + |x|^\alpha)k(t, x, y).$$

Adapting the arguments used in [5] and [11], we show the following intrinsic ultracontractivity

$$k_\mu(t, x, y) \leq c_1 e^{\lambda_0 t} e^{c_2 t^{-b}} \psi(x) \psi(y), \quad t > 0, x, y \in \mathbb{R}^N,$$

where c_1, c_2 are positive constant, $b = \frac{\beta-\alpha+2}{\beta+\alpha-2}$, provided that $N > 2$, $\alpha \geq 2$ and $\beta > \alpha - 2$. So one deduces the heat kernel estimate

$$k(t, x, y) \leq c_1 e^{\lambda_0 t} e^{c_2 t^{-b}} \times \left(|x|^{-\frac{N-1}{2}-\frac{\beta-\alpha}{4}} e^{-\frac{2}{\beta-\alpha+2}|x|^{\frac{\beta-\alpha+2}{2}}} \right) \left(\frac{|y|^{-\frac{N-1}{2}-\frac{\beta-\alpha}{4}} e^{-\frac{2}{\beta-\alpha+2}|y|^{\frac{\beta-\alpha+2}{2}}}}{1 + |y|^\alpha} \right)$$

for $t > 0, |x|, |y| \geq 1$. As an application we obtain the behaviour of all eigenfunctions of A_p at infinity. With respect to t we prove the following sharp estimates

$$k_\mu(t, x, y) \leq C t^{-\frac{N}{2}} (1 + |x|^\alpha)^{\frac{2-N}{4}} (1 + |y|^\alpha)^{\frac{2-N}{4}}$$

for $0 < t \leq 1$ and $x, y \in \mathbb{R}^N$. Here we use the results in [14] and weighted Nash inequalities introduced in [1]. We end this section by giving a brief description of how to extend the heat kernel estimates to a more general class of elliptic operators in divergence form.

In the sequel we denote by $B_R \subset \mathbb{R}^N$ the open ball, centered at 0 with radius $R > 0$.

2 Estimate of the ground state ψ

We begin by estimating the eigenfunction ψ corresponding to the largest eigenvalue λ_0 of A . First we recall some spectral properties obtained in [3] and [11].

Proposition 2.1 *Assume $N > 2$, $\alpha \geq 2$ and $\beta > \alpha - 2$ then*

- (i) *the resolvent of A_p is compact in $L^p(\mathbb{R}^N)$;*
- (ii) *the spectrum of A_p consists of a sequence of negative real eigenvalues which accumulates at $-\infty$. Moreover, $\sigma(A_p)$ is independent of p ;*

(iii) the semigroup $T_p(\cdot)$ is irreducible, the eigenspace corresponding to the largest eigenvalue λ_0 of A_p is one-dimensional and is spanned by strictly positive functions ψ , which is radial, belongs to $C_b^{1+\nu}(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$ for any $\nu \in (0, 1)$ and tends to 0 when $|x| \rightarrow \infty$.

We can now prove upper and lower estimates for ψ . We note here that the proof of [11, Proposition 3.1] cannot be adapted to our situation. So, we propose to use another technique to estimate ψ .

Proposition 2.2 *Let $\lambda_0 < 0$ be the largest eigenvalue of A_p and ψ be the corresponding eigenfunction. If $N > 2$, $\alpha \geq 2$ and $\beta > \alpha - 2$ then*

$$C_1 |x|^{-\frac{N-1}{2} - \frac{\beta-\alpha}{4}} e^{-\frac{2}{\beta-\alpha+2}|x|^{\frac{\beta-\alpha+2}{2}}} \leq \psi(x) \leq C_2 |x|^{-\frac{N-1}{2} - \frac{\beta-\alpha}{4}} e^{-\frac{2}{\beta-\alpha+2}|x|^{\frac{\beta-\alpha+2}{2}}}$$

for any $x \in \mathbb{R}^N \setminus B_1$ and some positive constants C_1, C_2 .

Proof: Since the eigenfunction is radial, we have to study the asymptotic behavior of the solution of an ordinary differential equation. We follow the idea of the WKB method (see [17]), but since the error function is not bounded we need to compute it directly.

Let $f_{\alpha,\beta,\lambda}$ be the function

$$f_{\alpha,\beta,\lambda}(x) = |x|^{-\frac{N-1}{2}} h^{-\frac{1}{4}}(|x|) \exp \left\{ - \int_R^{|x|} h^{\frac{1}{2}}(s) ds - \int_R^{|x|} v_\lambda(s) ds \right\}, \quad (1)$$

where $\lambda \in \mathbb{R}$, $h(r) = \frac{r^\beta}{1+r^\alpha}$, and v_λ is a smooth function to be chosen later on. If we set

$$w(r) = r^{\frac{N-1}{2}} f_{\alpha,\beta,\lambda}(r), \quad (2)$$

then

$$w' = w \left(-\frac{h'}{4h} - h^{\frac{1}{2}} - v_\lambda \right) \quad \text{and} \quad w'' = w(g + m + h), \quad (3)$$

where

$$g = \frac{5}{16} \left(\frac{h'}{h} \right)^2 - \frac{h''}{4h} + v_\lambda^2 + v_\lambda \left(\frac{h'}{2h} + 2h^{\frac{1}{2}} \right) - v_\lambda' - m \quad (4)$$

and

$$m(r) := \frac{(N-1)(N-3)}{4r^2}.$$

On the other hand, taking in mind (2) we also obtain

$$w''(r) = r^{\frac{N-1}{2}} \left(f_{\alpha,\beta,\lambda}'' + \frac{N-1}{r} f_{\alpha,\beta,\lambda}' + \frac{(N-1)(N-3)}{4r^2} f_{\alpha,\beta,\lambda} \right). \quad (5)$$

Comparing (3) and (5) we get

$$f_{\alpha,\beta,\lambda}'' + \frac{N-1}{r} f_{\alpha,\beta,\lambda}' = \frac{r^\beta}{1+r^\alpha} f_{\alpha,\beta,\lambda} + g f_{\alpha,\beta,\lambda}.$$

That is

$$\Delta f_{\alpha,\beta,\lambda}(x) - \frac{|x|^\beta}{1+|x|^\alpha} f_{\alpha,\beta,\lambda}(x) = g(|x|) f_{\alpha,\beta,\lambda}(x). \quad (6)$$

To evaluate the function g we set $\xi = \frac{\beta-\alpha}{2} + 1$, which is positive by the condition $\beta > \alpha - 2$. We have

$$\frac{h'}{h} = \frac{1}{r}(\beta - \alpha) + \frac{1}{r}O(r^{-\alpha}), \quad \frac{h''}{h} = \frac{1}{r^2}(\beta - \alpha)(\beta - \alpha - 1) + \frac{1}{r^2}O(r^{-\alpha}).$$

Then (4) is reduced to

$$\begin{aligned}
g(r) &= -v'_\lambda + \frac{v_\lambda}{r} \left(\xi - 1 + O(r^{-\alpha}) + 2r^\xi \sqrt{\frac{r^\alpha}{1+r^\alpha}} \right) + v_\lambda^2 \\
&\quad + \frac{c_0}{r^2} + \frac{1}{r^2} (O(r^{-\alpha}) + O(r^{-2\alpha})) \\
&= -v'_\lambda + \frac{v_\lambda}{r} \left(\xi - 1 + O(r^{-\alpha}) + 2r^\xi - 2r^\xi \frac{(1+r^\alpha)^{1/2} - r^{\alpha/2}}{(1+r^\alpha)^{1/2}} \right) \\
&\quad + v_\lambda^2 + \frac{c_0}{r^2} + \frac{1}{r^2} O(r^{-\alpha}) \\
&= -v'_\lambda + \frac{v_\lambda}{r} \left(\xi - 1 + 2r^\xi + (1+r^\xi)O(r^{-\alpha}) \right) + v_\lambda^2 \\
&\quad + \frac{c_0}{r^2} + \frac{1}{r^2} O(r^{-\alpha}), \tag{7}
\end{aligned}$$

where $c_0 = c_0(\xi) = \left(\frac{\xi-1}{2}\right)^2 + \frac{\xi-1}{2} - \frac{(N-1)(N-3)}{4}$. So, if we take in (7)

$$v_\lambda(r) = \sum_{i=1}^k c_i \frac{1}{r^{i\xi+1}}, \quad r \geq 1,$$

we obtain

$$\begin{aligned}
r^2 g(r) &= \sum_{i=1}^k c_i (i\xi + 1) \frac{1}{r^{i\xi}} + (\xi - 1) \sum_{i=1}^k c_i \frac{1}{r^{i\xi}} + 2 \sum_{i=0}^{k-1} c_{i+1} \frac{1}{r^{i\xi}} \\
&\quad + \left(\sum_{i=1}^k c_i \frac{1}{r^{i\xi}} + \sum_{i=0}^{k-1} c_{i+1} \frac{1}{r^{i\xi}} \right) O(r^{-\alpha}) \\
&\quad + \sum_{i,j=1}^k c_i c_j \frac{1}{r^{(i+j)\xi}} + c_0 + O(r^{-\alpha}) \\
&= \sum_{i=2}^{k-1} \left[c_i \xi (i+1) + 2c_{i+1} + \sum_{j+s=i} c_j c_s \right] \frac{1}{r^{i\xi}} + (2c_1 \xi + 2c_2) r^{-\xi} \\
&\quad + c_k \xi (k+1) \frac{1}{r^{k\xi}} + 2c_1 + \sum_{i+j \geq k} \frac{c_i c_j}{r^{(i+j)\xi}} + c_0 + O(r^{-\alpha}).
\end{aligned}$$

We can choose c_1, \dots, c_k such that

$$2c_1 + c_0 = \lambda, \quad 2c_1 \xi + 2c_2 = 0 \quad \text{and} \quad \left[\xi(i+1)c_i + 2c_{i+1} + \sum_{j+s=i} c_j c_s \right] = 0$$

for $i = 2, \dots, k-1$ and obtain

$$r^2 g(r) = \lambda + c_k \xi (k+1) \frac{1}{r^{k\xi}} + \sum_{i+j \geq k} \frac{c_i c_j}{r^{(i+j)\xi}} + O(r^{-\alpha}). \tag{8}$$

Thus,

$$g(r) = O\left(\frac{1}{r^{k\xi+2}}\right) + O\left(\frac{1}{r^{\alpha+2}}\right) + \frac{\lambda}{r^2}.$$

Since $\xi > 0$ there exists $k \in \mathbb{N}$ such that $k\xi + 2 - \alpha > 0$. So we have

$$(1 + |x|^\alpha) \Delta f_{\alpha, \beta, \lambda}(x) - |x|^\beta f_{\alpha, \beta, \lambda}(x) = o(1) f_{\alpha, \beta, \lambda}(x) + \lambda \frac{1 + |x|^\alpha}{|x|^2} f_{\alpha, \beta, \lambda}(x). \tag{9}$$

We prove first the upper bound.

For ψ we know that

$$\Delta\psi - \frac{|x|^\beta}{1+|x|^\alpha}\psi - \frac{\lambda_0}{1+|x|^\alpha}\psi = 0. \quad (10)$$

Since $\alpha-2 \geq 0$ and $\lambda_0 < 0$, for $|x|$ large enough we have $o(1) + 2\lambda_0 \frac{1+|x|^\alpha}{|x|^2} < \lambda_0$. Then, by (9), it follows that

$$(1+|x|^\alpha)\Delta f_{\alpha,\beta,2\lambda_0}(x) - |x|^\beta f_{\alpha,\beta,2\lambda_0}(x) < \lambda_0 f_{\alpha,\beta,2\lambda_0}(x).$$

Thus,

$$\Delta f_{\alpha,\beta,2\lambda_0}(x) - \frac{|x|^\beta}{1+|x|^\alpha} f_{\alpha,\beta,2\lambda_0}(x) - \frac{\lambda_0}{1+|x|^\alpha} f_{\alpha,\beta,2\lambda_0}(x) < 0, \quad (11)$$

in $\mathbb{R}^N \setminus B_R$ for some $R > 0$. Comparing (10) and (11), in $\mathbb{R}^N \setminus B_R$ we have

$$\Delta(f_{\alpha,\beta,2\lambda_0} - C\psi) < \frac{\lambda_0 + |x|^\beta}{1+|x|^\alpha} (f_{\alpha,\beta,2\lambda_0} - C\psi) \quad \text{for any constant } C > 0.$$

Since $\beta > 0$, we have

$$W(x) := \frac{\lambda_0 + |x|^\beta}{1+|x|^\alpha} > 0$$

for $|x|$ large enough. Since both $f_{\alpha,\beta,2\lambda_0}$ and ψ tend to 0 as $|x| \rightarrow \infty$ and since there exists C_2 such that $\psi \leq C_2 f_{\alpha,\beta,2\lambda_0}$ on ∂B_R , we can apply the maximum principle to the problem

$$\begin{cases} \Delta g(x) - W(x)g(x) < 0 & \text{in } \mathbb{R}^N \setminus B_R, \\ g(x) \geq 0 & \text{in } \partial B_R, \\ \lim_{|x| \rightarrow \infty} g(x) = 0, \end{cases}$$

where $g := f_{\alpha,\beta,2\lambda_0} - C_2^{-1}\psi$, to obtain $\psi \leq C_2 f_{\alpha,\beta,2\lambda_0}$ in $\mathbb{R}^N \setminus B_R$. Here one has to note that since $\lim_{|x| \rightarrow \infty} g(x) = 0$, one can see that the classical maximum principle on bounded domains can be applied, cf. [8, Theorem 3.5]. Then,

$$\begin{aligned} \psi(x) &\leq C_2 |x|^{-\frac{N-1}{2} - \frac{1}{4}(\beta-\alpha)} \exp \left\{ - \int_R^{|x|} \sqrt{\frac{r^\beta}{1+r^\alpha}} dr \right\} \exp \left\{ - \int_R^{|x|} v_{2\lambda_0}(r) dr \right\} \\ &\leq C_3 |x|^{-\frac{N-1}{2} - \frac{1}{4}(\beta-\alpha)} \exp \left\{ - \int_R^{|x|} \sqrt{\frac{r^\beta}{1+r^\alpha}} dr \right\}, \end{aligned}$$

since

$$\begin{aligned} \lim_{|x| \rightarrow \infty} \int_R^{|x|} v_\lambda(r) dr &= \lim_{|x| \rightarrow \infty} \sum_{j=1}^k \frac{c_j}{j^\xi} (R^{-j\xi} - |x|^{-j\xi}) \\ &= \sum_{j=1}^k \frac{c_j}{j^\xi} R^{-j\xi}. \end{aligned} \quad (12)$$

As regards lower bounds of ψ , we observe that, from (9), we have

$$\Delta f_{\alpha,\beta,0}(x) - \frac{|x|^\beta}{1+|x|^\alpha} f_{\alpha,\beta,0}(x) = \frac{o(1)}{1+|x|^\alpha} f_{\alpha,\beta,0}(x) > \frac{\lambda_0}{1+|x|^\alpha} f_{\alpha,\beta,0}(x)$$

if $|x| \geq R$ for some suitable $R > 0$. Then,

$$\Delta f_{\alpha,\beta,0}(x) > \frac{|x|^\beta}{1+|x|^\alpha} f_{\alpha,\beta,0}(x) + \frac{\lambda_0}{1+|x|^\alpha} f_{\alpha,\beta,0}(x)$$

Since $\frac{\lambda_0}{1+|x|^\alpha} \psi = \Delta \psi(x) - \frac{|x|^\beta}{1+|x|^\alpha} \psi$ we have

$$\Delta(f_{\alpha,\beta,0} - \psi) > \frac{|x|^\beta + \lambda_0}{1+|x|^\alpha} (f_{\alpha,\beta,0} - \psi).$$

We can assume that $|x|^\beta + \lambda_0$ is positive for $|x| \geq R$ and, arguing as above, by the maximum principle and using (12) we get

$$\psi(x) \geq C_1 f_{\alpha,\beta,0}(x) \geq C_1 |x|^{-\frac{N-1}{2} - \frac{1}{4}(\beta-\alpha)} \exp \left\{ - \int_R^{|x|} \sqrt{\frac{r^\beta}{1+r^\alpha}} dr \right\}$$

for $|x| \geq R$. Since $0 < \psi \in C(\mathbb{R}^N)$, by changing the constants, the above upper and lower estimates remain valid for $1 \leq |x| \leq R$. This ends the proof of the proposition. \square

3 Intrinsic ultracontractivity and heat kernel estimates

Let us now introduce on $L_\mu^2 := L^2(\mathbb{R}^N, d\mu)$ the bilinear form

$$a_\mu(u, v) = \int_{\mathbb{R}^N} \nabla u \cdot \nabla \bar{v} dx + \int_{\mathbb{R}^N} V u \bar{v} d\mu, \quad u, v \in D(a_\mu), \quad (13)$$

where $V(x) = |x|^\beta$, $d\mu(x) = (1 + |x|^\alpha)^{-1} dx$ and $D(a_\mu) = \overline{C_c^\infty(\mathbb{R}^N)}^{\|\cdot\|_H}$ with H the Hilbert space

$$H = \{u \in L_\mu^2 : V^{1/2} u \in L_\mu^2, \nabla u \in (L^2(\mathbb{R}^N))^N\}$$

endowed with the inner product

$$\langle u, v \rangle_H = \int_{\mathbb{R}^N} (1 + V) u \bar{v} d\mu + \int_{\mathbb{R}^N} \nabla u \cdot \nabla \bar{v} dx.$$

Since a_μ is a closed, symmetric and accretive form, to a_μ we associate the self-adjoint operator A_μ defined by

$$D(A_\mu) = \left\{ u \in D(a_\mu) : \exists g \in L_\mu^2 \text{ s.t. } a_\mu(u, v) = - \int_{\mathbb{R}^N} g \bar{v} d\mu, \forall v \in D(a_\mu) \right\},$$

$$A_\mu u = g,$$

see e.g., [18, Prop. 1.24]. By general results on positive self-adjoint operators induced by nonnegative quadratic forms in Hilbert spaces (see e.g., [18, Prop. 1.51, Thms. 1.52, 2.6, 2.13]) A_μ generates a positive analytic semigroup $(e^{tA_\mu})_{t \geq 0}$ in L_μ^2 .

We need to show that the semigroup e^{tA_μ} coincides with the semigroup $T_p(\cdot)$ generated by A_p in $L^p(\mathbb{R}^N)$ on $L^p(\mathbb{R}^N) \cap L_\mu^2$.

Lemma 3.1 *We have*

$$D(A_\mu) = \left\{ u \in D(a_\mu) \cap W_{loc}^{2,2}(\mathbb{R}^N) : (1 + |x|^\alpha) \Delta u - V(x) u \in L_\mu^2 \right\}$$

and $A_\mu u = (1 + |x|^\alpha) \Delta u - V(x) u$ for $u \in D(A_\mu)$. Moreover, if $\lambda > 0$ and $f \in L^p(\mathbb{R}^N) \cap L_\mu^2$, then

$$(\lambda - A_\mu)^{-1} f = (\lambda - A_p)^{-1} f.$$

Proof: The inclusion " \subset " is obtained, taking $v \in C_c^\infty(\mathbb{R}^N)$ in (13), by local elliptic regularity. As regards the inclusion " \supset " we consider $u \in D(a_\mu) \cap W_{loc}^{2,2}(\mathbb{R}^N)$ such that $g := (1 + |x|^\alpha)\Delta u - V(x)u \in L_\mu^2$ and consider $v \in C_c^\infty(\mathbb{R}^N)$. Integrating by parts we obtain

$$a_\mu(u, v) = - \int g v d\mu. \quad (14)$$

By the density of $C_c^\infty(\mathbb{R}^N)$ in $D(a_\mu)$ we have equation (14) for every $v \in D(a_\mu)$. This implies that $u \in D(A_\mu)$.

To show the coherence of the resolvent, we consider $f \in C_c^\infty(\mathbb{R}^N)$ and let $u = (\lambda - A)^{-1}f$. Since $f \in L^2(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$, by [3, Theorem 3.7] and [3, Theorem 4.4], it follows that $u \in D_2(A)$. So, we have $\nabla u \in L^2(\mathbb{R}^N)$ and $Vu \in L^2(\mathbb{R}^N)$. Moreover, it is clear that $u \in L_\mu^2$, and

$$\begin{aligned} \|V^{1/2}u\|_{L_\mu^2}^2 &\leq \int_{\mathbb{R}^N} V(x)u^2 dx \\ &\leq \int_{B(1)} u^2 dx + \int_{\mathbb{R}^N \setminus B(1)} V^2(x)u^2 dx \\ &\leq \|u\|_2^2 + \|Vu\|_2^2. \end{aligned} \quad (15)$$

This yields $u \in H$. Since $C_c^\infty(\mathbb{R}^N)$ is dense in $D_2(A)$, see [3, Lemma 4.3], we can find a sequence $(u_n) \subset C_c^\infty(\mathbb{R}^N)$ such that u_n converges to u in the operator norm. Then, u_n converges to u in $L^2(\mathbb{R}^N)$ and hence in L_μ^2 . By [3, Lemma 4.2] ∇u_n converges to ∇u in $L^2(\mathbb{R}^N)$ and hence in L_μ^2 . Finally replacing u with $u_n - u$ in (15) we have that $V^{1/2}u_n$ converges to $V^{1/2}u$ in L_μ^2 . Thus we have proved that $u \in D(a_\mu)$. Integration by parts we obtain

$$a(u, v) = -(\lambda u - f, v)_{L_\mu^2}.$$

That is $u \in D(A_\mu)$ and $\lambda u - A_\mu u = f$. Therefore, $(\lambda - A_\mu)^{-1}f = (\lambda - A_p)^{-1}f$ for all $f \in C_c^\infty(\mathbb{R}^N)$ and so by density the last statement follows. \square

The previous Lemma implies in particular that

$$e^{tA_\mu} f = T_p(t)f = \int_{\mathbb{R}^N} k(t, x, y)f(y) dy, \quad f \in L^p(\mathbb{R}^N) \cap L_\mu^2.$$

By density we obtain that the semigroup e^{tA_μ} admits the integral representation $e^{tA_\mu} f(x) = \int_{\mathbb{R}^N} k_\mu(t, x, y)f(y)d\mu(y)$ for all $f \in L_\mu^2$, where

$$k_\mu(t, x, y) = (1 + |y|^\alpha)k(t, x, y), \quad t > 0, x, y \in \mathbb{R}^N. \quad (16)$$

Let us now give the first application of Proposition 2.2. The proof is similar to the one given in [11, Proposition 3.4] and is based on the semigroup law and the symmetry of $k_\mu(t, \cdot, \cdot)$ for $t > 0$.

Proposition 3.2 *If $N > 2$, $\alpha \geq 2$ and $\beta > \alpha - 2$, then*

$$k(t, x, x) \geq M e^{\lambda_0 t} \left(|x|^{\frac{\alpha-\beta}{4} - \frac{N-1}{2}} e^{-\frac{2}{\beta-\alpha+2}|x|^{\frac{\beta-\alpha+2}{2}}} \right)^2 (1 + |x|^\alpha)^{-1}, \quad t > 0,$$

for all $x \in \mathbb{R}^N \setminus B_1$ and some constant $M > 0$.

We now give the main result of this section.

Theorem 3.3 *If $N > 2$, $\alpha \geq 2$ and $\beta > \alpha - 2$ then*

$$k(t, x, y) \leq c_1 e^{\lambda_0 t + c_2 t^{-b}} |x|^{-\frac{N-1}{2} - \frac{\beta-\alpha}{4}} e^{-\frac{2}{\beta-\alpha+2}|x|^{\frac{\beta-\alpha+2}{2}}} \frac{|y|^{-\frac{N-1}{2} - \frac{\beta-\alpha}{4}}}{1 + |y|^\alpha} e^{-\frac{2}{\beta-\alpha+2}|y|^{\frac{\beta-\alpha+2}{2}}}$$

for $t > 0$, $x, y \in \mathbb{R}^N \setminus B_1$, where c_1, c_2 are positive constants and $b = \frac{\beta-\alpha+2}{\beta+\alpha-2}$.

Proof: Let us prove first

$$k(t, x, y) \leq c_1 e^{c_2 t^{-b}} |x|^{-\frac{N-1}{2} - \frac{\beta-\alpha}{4}} e^{-\frac{2}{\beta-\alpha+2}|x|^{\frac{\beta-\alpha+2}{2}}} \frac{|y|^{-\frac{N-1}{2} - \frac{\beta-\alpha}{4}}}{1 + |y|^\alpha} e^{-\frac{2}{\beta-\alpha+2}|y|^{\frac{\beta-\alpha+2}{2}}} \quad (17)$$

for $0 < t \leq 1$, $x, y \in \mathbb{R}^N \setminus B_1$. By adapting the arguments used in [5, Subsect. 4.4 and 4.5], we have only to show the following estimates

$$\int_{\mathbb{R}^N} g|u|^2 d\mu \leq C \|g\|_{L_\mu^{N/2}} a_\mu(u, u), \quad u \in D(a_\mu), \quad g \in L_\mu^{N/2}, \quad (18)$$

and

$$\int_{\mathbb{R}^N} -\log \psi |u|^2 d\mu \leq \varepsilon a_\mu(u, u) + (C_1 \varepsilon^{-b} + C_2) \|u\|_{L_\mu^2}^2, \quad u \in D(a_\mu). \quad (19)$$

To prove (18) we observe that using Hölder and Sobolev inequality we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} g|u|^2 d\mu &\leq C \left(\int_{\mathbb{R}^N} |g|^{N/2} d\mu \right)^{2/N} \left(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}} d\mu \right)^{\frac{N-2}{N}} \\ &= C \|g\|_{L_\mu^{N/2}} \|u\|_{L_\mu^{2^*}}^2 \\ &\leq C \|g\|_{L_\mu^{N/2}} \|\nabla u\|_2^2 \leq C \|g\|_{L_\mu^{N/2}} a_\mu(u, u), \quad u \in D(a_\mu). \end{aligned}$$

To show (19), we apply the lower estimate of ψ obtained in Proposition 2.2

$$\begin{aligned} -\log \psi &\leq -\left(\log C_1 - \frac{2}{\beta-\alpha+2} \right) + \frac{2N-2+\beta-\alpha}{4} \log |x| \\ &\quad + \frac{2}{\beta-\alpha+2} |x|^{\frac{\beta-\alpha+2}{2}} \end{aligned}$$

for $|x| \geq 1$. Hence, there are positive constants C_1, C_2 such that

$$-\log \psi \leq C_1 |x|^{\frac{\beta-\alpha+2}{2}} + C_2, \quad x \in \mathbb{R}^N.$$

Since $\xi = \frac{\beta-\alpha}{2} + 1 < \beta$ we have

$$|x|^\xi \leq \varepsilon |x|^\beta + C \varepsilon^{-\frac{\xi}{\beta-\xi}} = \varepsilon V(x) + C \varepsilon^{-b}$$

for all $\varepsilon > 0$. Thus,

$$-\log \psi \leq \varepsilon V + c_1 \varepsilon^{-b} + c_2.$$

Taking into account that $\int_{\mathbb{R}^N} V|u|^2 d\mu \leq a_\mu(u, u)$ for all $u \in D(a_\mu)$, we obtain (19). This ends the proof of (17).

It remains to prove that

$$k(t, x, y) \leq C e^{\lambda_0 t} |x|^{-\frac{N-1}{2} - \frac{\beta-\alpha}{4}} e^{-\frac{2}{\beta-\alpha+2}|x|^{\frac{\beta-\alpha+2}{2}}} \frac{|y|^{-\frac{N-1}{2} - \frac{\beta-\alpha}{4}}}{1 + |y|^\alpha} e^{-\frac{2}{\beta-\alpha+2}|y|^{\frac{\beta-\alpha+2}{2}}} \quad (20)$$

for $t > 1$, $x, y \in \mathbb{R}^N \setminus B_1$ and some constant $C > 0$. To this purpose we use the semigroup law and the symmetry of $k_\mu(t, \cdot, \cdot)$ to infer that

$$k_\mu(t, x, y) = \int_{\mathbb{R}^N} k_\mu(t - 1/2, x, z) k_\mu(1/2, y, z) d\mu(z), \quad t > 1/2, \quad x, y \in \mathbb{R}^N.$$

By (17), the function $k_\mu(1/2, y, \cdot)$ belongs to L_μ^2 . Hence,

$$k_\mu(t, x, y) = (e^{(t-\frac{1}{2})A_\mu} k_\mu(1/2, y, \cdot))(x), \quad t > 1/2, \quad x, y \in \mathbb{R}^N.$$

Using again the semigroup law and the symmetry we deduce that

$$\begin{aligned} k_\mu(t, x, x) &= \int_{\mathbb{R}^N} |k_\mu(t/2, x, y)|^2 d\mu(y) \\ &\leq M e^{\lambda_0(t-1)} \|k_\mu(1/2, x, \cdot)\|_{L_\mu^2}^2 \\ &= M e^{\lambda_0(t-1)} k_\mu(1, x, x). \end{aligned}$$

So, by applying (17) to $k_\mu(1, x, x)$ and using the inequality

$$k_\mu(t, x, y) \leq (k_\mu(t, x, x))^{1/2} (k_\mu(t, y, y))^{1/2},$$

one obtains (20). \square

Remark 3.4 *It follows from Proposition 3.2 that the estimates obtained for the heat kernel k in Theorem 3.3 could be sharp in the space variables but certainly not in the time variable as we will prove in Proposition 3.8.*

Remark 3.5 *In the above proof we use the Sobolev inequality*

$$\|u\|_{L_\mu^{2^*}}^2 \leq C \|\nabla u\|_2^2$$

which holds in $D(a_\mu)$ but not in H (consider for example the case where $\alpha > \beta + N$ and $u = 1$).

As a consequence of Theorem 3.3 we deduce some estimates for the eigenfunctions.

Corollary 3.6 *If the assumptions of Theorem 3.3 hold, then all normalized eigenfunctions ψ_j of A_2 satisfy*

$$|\psi_j(x)| \leq C_j |x|^{\frac{\alpha-\beta}{4} - \frac{N-1}{2}} e^{-\frac{2}{\beta-\alpha+2}|x|^{\frac{\beta-\alpha+2}{2}}},$$

for all $x \in \mathbb{R}^N \setminus B_1$, $j \in \mathbb{N}$ and a constant $C_j > 0$.

Proof: Let λ_j be an eigenvalue of A_2 and denote by ψ_j any normalized (i.e. $\|\psi_j\|_{L^2(\mathbb{R}^N)} = 1$) eigenfunction associated to λ_j . Then, as in the proof of Theorem 3.3, we have

$$\begin{aligned} e^{\lambda_j t} |\psi_j(x)| &= \left| \int_{\mathbb{R}^N} k_\mu(t, x, y) \psi_j(y) d\mu(y) \right| \\ &\leq \left(\int_{\mathbb{R}^N} k_\mu(t, x, y)^2 d\mu(y) \right)^{\frac{1}{2}} \|\psi_j\|_{L_\mu^2} \\ &= (k_\mu(2t, x, x))^{\frac{1}{2}}, \end{aligned}$$

for $t > 0$ and $x \in \mathbb{R}^N$. So, the estimates follow from Theorem 3.3. \square

Remark 3.7 It is possible to obtain better estimates of the kernels k with respect to the time variable t for small t . In fact if we denote by $S(\cdot)$ the semigroup generated by $(1 + |x|^\alpha)\Delta$ in $C_b(\mathbb{R}^N)$, which is given by a kernel p , then by domination we have $0 < k(t, x, y) \leq p(t, x, y)$ for $t > 0$ and $x, y \in \mathbb{R}^N$. So, by [14, Theorem 2.6 and Theorem 2.14], it follows that

$$\begin{aligned} k(t, x, y) &\leq Ct^{-N/2}(1 + |x|)^{2-N}(1 + |y|)^{2-N-\alpha}, \quad \alpha > 4, \\ k(t, x, y) &\leq Ct^{-N/2}(1 + |x|^\alpha)^{\frac{2-N}{4}}(1 + |y|^\alpha)^{\frac{2-N}{4}-1}, \quad 2 < \alpha \leq 4 \end{aligned} \quad (21)$$

for $0 < t \leq 1$, $x, y \in \mathbb{R}^N$.

Using a domination argument and [14, Proposition 2.10] we can improve the estimate (21).

Proposition 3.8 If $\alpha \geq 2$, $\beta > \alpha - 2$ and $N > 2$, then the kernel k_μ satisfies

$$k_\mu(t, x, y) \leq Ct^{-N/2}(1 + |x|^\alpha)^{\frac{2-N}{4}}(1 + |y|^\alpha)^{\frac{2-N}{4}}$$

for $0 < t \leq 1$ and $x, y \in \mathbb{R}^N$.

Proof: It suffices to consider the case $\alpha > 4$.

By domination one sees that weighted Nash inequalities given in [14, Proposition 2.10] hold for the quadratic form a_μ . Hence, by [1, Corollary 2.8], the results is proved provided that the function $\varphi(x) = (1 + |x|^\alpha)^{\frac{2-N}{4}}$ is a Lyapunov function in the sense of [14, Definition 2.1].

A simple computation yields

$$\begin{aligned} A\varphi &= \left[\gamma(N + \alpha - 2)|x|^{\alpha-2} + \gamma(\gamma - \alpha)|x|^{\alpha-2} \frac{|x|^\alpha}{1 + |x|^\alpha} \right] \varphi(x) \\ &= \left[\gamma(\gamma + N - 2) \frac{|x|^{2\alpha-2}}{1 + |x|^\alpha} + \gamma(\alpha - 2 + N) \frac{|x|^{\alpha-2}}{1 + |x|^\alpha} - |x|^\beta \right] \varphi(x) \\ &\leq \left[\gamma(\gamma + N - 2) \frac{|x|^{2\alpha-2}}{1 + |x|^\alpha} - |x|^\beta \right] \varphi(x), \end{aligned}$$

where $\gamma := \frac{\alpha(2-N)}{4}$. We note that $\gamma < 2 - N$, since $\alpha > 4$. Now, using the fact that $\beta > \alpha - 2$, we deduce that

$$\gamma(\gamma + N - 2) \frac{|x|^{2\alpha-2}}{1 + |x|^\alpha} \leq \gamma(\gamma + N - 2) |x|^{\alpha-2} \leq |x|^\beta + \kappa$$

for some $\kappa > 0$. Thus, $A\varphi \leq \kappa\varphi$. Using the same arguments as in [14, Lemma 2.13] we obtain that φ is Lyapunov function for A . \square

As in [11] heat kernel estimates can be also obtained for a more general class of elliptic operators.

Let us consider the operator B , defined on smooth functions u by

$$Bu = (1 + |x|^\alpha) \sum_{j,k=1}^N D_k(a_{kj} D_j u) - Wu,$$

under the following set of assumptions:

Hypotheses 1

1. the coefficients $a_{kj} = a_{jk}$ belong to $C_b(\mathbb{R}^N) \cap W_{\text{loc}}^{1,\infty}(\mathbb{R}^N)$ for any $j, k = 1, \dots, N$ and there exists a positive constant η such that

$$\eta|\xi|^2 \leq \sum_{j,k=1}^N a_{kj}(x) \xi_k \xi_j, \quad x, \xi \in \mathbb{R}^N;$$

2. $W \in L^1_{\text{loc}}(\mathbb{R}^N)$ satisfies $W(x) \geq |x|^\beta$ for any $x \in \mathbb{R}^N$ and some $\beta > \alpha - 2$;
3. $\alpha \geq 2$ and $D_j a_{kj}(x) = o(|x|^{\frac{\beta-\alpha}{2}})$ as $|x| \rightarrow \infty$.

On L^2_μ we define the bilinear form

$$b_\mu(u, v) = \sum_{j,k=1}^N \int_{\mathbb{R}^N} a_{kj} D_k u D_j \bar{v} dx + \int_{\mathbb{R}^N} W u \bar{v} d\mu, \quad u, v \in D(b_\mu),$$

where $D(b_\mu) = \overline{C_c^\infty(\mathbb{R}^N)}^{\|\cdot\|_{\mathcal{H}}}$ with \mathcal{H} the Hilbert space

$$\mathcal{H} = \{u \in L^2_\mu : W^{1/2}u \in L^2_\mu, \nabla u \in (L^2(\mathbb{R}^N))^N\}.$$

Since b_μ is a symmetric, accretive and closable form, we can associate a positive strongly continuous semigroup $S_\mu(\cdot)$ in L^2_μ . The same arguments as in the beginning of this section show that the infinitesimal generator B_μ of this semigroup is the realization in L^2_μ of the operator B with domain $D(B_\mu) = \{u \in D(b_\mu) \cap W^{2,2}_{\text{loc}}(\mathbb{R}^N) : Bu \in L^2_\mu\}$. Let us denote by p_μ the heat kernel associated to $S_\mu(\cdot)$.

We will also need the bilinear form

$$a_{\mu,\theta}(u, v) = \int_{\mathbb{R}^N} \nabla u \cdot \nabla \bar{v} dx + \theta^2 \int_{\mathbb{R}^N} V u \bar{v} d\mu, \quad u, v \in D(a_{\mu,\theta}) = D(a_\mu).$$

The same arguments as in the proof of Theorem 3.3 can be used to show that the kernel $k_{\mu,\theta}$ of the analytic semigroup associated to the form $a_{\mu,\theta}$ in L^2_μ satisfies

$$0 < k_{\mu,\theta}(t, x, y) \leq K_\theta e^{\lambda_{0,\theta} t} e^{\tilde{c}_\theta t^{-b}} \psi_\theta(x) \psi_\theta(y), \quad t > 0, x, y \in \mathbb{R}^N, \quad (22)$$

where \tilde{c}_θ and K_θ are positive constants, $\lambda_{0,\theta}$ is the largest (negative) eigenvalue of the minimal realization of operator $A_\theta := (1 + |x|^\alpha)\Delta - \theta|x|^\beta$ in $L^2(\mathbb{R}^N)$, and ψ_θ is a corresponding positive and bounded eigenfunction. Moreover, there exist $C_{1,\theta}, C_{2,\theta} > 0$ such that

$$C_{1,\theta} \leq |x|^{-\frac{\alpha-\beta}{4} + \frac{N-1}{2}} e^{-\theta \frac{2}{\beta-\alpha+2}|x|} e^{\frac{\beta-\alpha+2}{2}|x|} \psi_\theta(x) \leq C_{2,\theta},$$

for any $x \in \mathbb{R}^N \setminus B_1$.

Using Theorem 3.3 and arguing as in [19] and [11, Theorem 3.9] we obtain the following heat kernel estimate.

Theorem 3.9 *Assume that Hypotheses 1 are satisfied and let*

$$\Lambda := \sup_{x, \xi \in \mathbb{R}^N \setminus \{0\}} |\xi|^{-2} \sum_{j,k=1}^N a_{kj}(x) \xi_k \xi_j.$$

Then, for any $\theta \in (0, \Lambda^{-1/2})$, we have

$$p_\mu(t, x, y) \leq M_\theta e^{\lambda_{0,\theta} t} e^{c_\theta t^{-b}} (|x||y|)^{\frac{\alpha-\beta}{4} - \frac{N-1}{2}} \times e^{-\theta \frac{2}{\beta-\alpha+2}|x|} e^{\frac{\beta-\alpha+2}{2}|x|} e^{-\theta \frac{2}{\beta-\alpha+2}|y|} e^{\frac{\beta-\alpha+2}{2}|y|}$$

for any $t > 0$ and $x, y \in \mathbb{R}^N \setminus B_1$, where M_θ, c_θ are positive constants, $b = \frac{\beta-\alpha+2}{\beta+\alpha-2}$, and $\lambda_{0,\theta}$ is the largest eigenvalue of the operator $(1 + |x|^\alpha)\Delta - \theta|x|^\beta$.

Proof: For the reader's convenience, we give the main ideas of the proof. Proving the above estimate is equivalent to showing that

$$\phi_\theta(x)^{-1} p_\mu(t, x, y) \phi_\theta(y)^{-1} \leq M_\theta e^{\lambda_{0,\theta} t} e^{c_\theta t^{-b}}, \quad t > 0, x, y \in \mathbb{R}^N, \quad (23)$$

where ϕ_θ is any smooth function satisfying

$$\phi_\theta(x) = |x|^{\frac{\alpha-\beta}{4} - \frac{N-1}{2}} e^{-\theta \frac{2}{\beta-\alpha+2} |x|^{\frac{\beta-\alpha+2}{2}}}, \quad x \in \mathbb{R}^N \setminus B_1.$$

If we denote by $T_{\phi_\theta} : L_{\phi_\theta^2 \mu}^2 \rightarrow L_\mu^2$ the isometry defined by $T_{\phi_\theta} f = \phi_\theta f$, then the left hand side of (23) is the kernel of the semigroup $(T_{\phi_\theta}^{-1} e^{tB_\mu} T_{\phi_\theta})_{t \geq 0}$ in $L_{\phi_\theta^2 \mu}^2$. It is clear that this semigroup is associated with the form $\tilde{b}_\mu(u, v) = b_\mu(\phi_\theta u, \phi_\theta v)$ for $u, v \in D(\tilde{b}_\mu) := \{u \in L_{\phi_\theta^2 \mu}^2 : \phi_\theta u \in D(b_\mu)\}$.

As in the proof of Theorem 3.3, it suffices to establish (23) for $t \in (0, 1]$. To this purpose one has to prove, as in the proof of [11, Theorem 3.9], the following assertions:

- (i) $\min\{u, 1\} \in D(\tilde{a}_{\mu,\theta})$ (resp. $D(\tilde{b}_\mu)$) for any nonnegative $u \in D(\tilde{a}_{\mu,\theta})$ (resp. $D(\tilde{b}_\mu)$);
- (ii) the semigroup $(T_{\phi_\theta}^{-1} e^{tB_\mu} T_{\phi_\theta})_{t \geq 0}$ and the semigroup $(T_{\phi_\theta}^{-1} e^{tA_{\mu,\theta}} T_{\phi_\theta})_{t \geq 0}$, associated to the form $\tilde{a}_{\mu,\theta} = a_{\mu,\theta}(\phi_\theta \cdot, \phi_\theta \cdot)$ with domain $D(\tilde{a}_{\mu,\theta}) = D(\tilde{b}_\mu)$, are positive, they map $L^\infty(\mathbb{R}^N)$ into itself and satisfy the estimates

$$\|T_{\phi_\theta}^{-1} e^{tB_\mu} T_{\phi_\theta}\|_{L(L^\infty(\mathbb{R}^N))} \leq e^{C_1 t}, \quad \|T_{\phi_\theta}^{-1} e^{tA_{\mu,\theta}} T_{\phi_\theta}\|_{L(L^\infty(\mathbb{R}^N))} \leq e^{C_1 t}, \quad t > 0,$$

for some positive constant C_1 ;

- (iii) the Log-Sobolev inequality

$$\int_{\mathbb{R}^N} u^2 (\log u) \phi_\theta^2 d\mu \leq \varepsilon \tilde{b}_\mu(u, u) + \|u\|_{L_{\phi_\theta^2 \mu}^2}^2 \log \|u\|_{L_{\phi_\theta^2 \mu}^2} + c_\theta (1 + \varepsilon^{-b}) \|u\|_{L_{\phi_\theta^2 \mu}^2}^2 \quad (24)$$

holds true for any nonnegative $u \in D(\tilde{b}_\mu) \cap L_{\phi_\theta^2 \mu}^1 \cap L^\infty(\mathbb{R}^N)$, where c_θ is the constant in (23).

So, applying (24) and combining [5, Lemma 2.1.2, Cor. 2.2.8 and Ex. 2.3.4], estimate (23) follows with $t \in (0, 1]$.

The proof of (i), (ii) and (iii) is similar to the one in [11, Theorem 3.9]. The proof of (iii) is based on the estimate $\tilde{b}_\mu(u, u) \geq \min\{\mu, \theta^{-1}\} \tilde{a}_{\mu,\theta}(u, u)$ which holds for any $u \in D(\tilde{b}_\mu) \subset D(\tilde{a}_{\mu,\theta})$, and (22). \square

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